Unit-2

[#] [=] 3×3 3×1

Matrix

25-08-23

→ Matrix is a kind of representation.

(Q.) Write down the eigen value & eqn of matrix

= Eigen value eqn of matrix,

$A\psi = \lambda I \psi$

$\Rightarrow A\psi - \lambda I \psi = 0$

$\Rightarrow (A - \lambda I)\psi = 0$ → B

$\psi \neq 0$ $\boxed{\text{Det.}(A - \lambda I) = 0}$

A = square matrix
ψ = eigen vector [column matrix]
I = unit matrix / Identity matrix
λ = eigen value [no]

(Q.) Find the eigen value and eigen vector of following matrix —

= (i) $A = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ ($\pm 1 = \psi$)

(i) $A - \lambda I$

$= \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 - 16 = 0$

or $(1-\lambda)^2 = 16$

or $1 - \lambda = \pm 4$

$\boxed{\lambda = 5, -3}$ 2×2 matrix ⇒ 2 eigen value

Note:

No. of eigen value = order of the matrix.

eigen vector for $\lambda_1 = -3$

We know that from eigen value eqn

$A\psi = \lambda I \psi$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

2×2 2×1 2×2 2×1

$\Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ 8x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ -3x_2 \end{bmatrix}$

2×1 2×1

no. of eq = no. of variable

here, no. of variable > no. of eqn (∴ ∞ no. soln)

$x_1 + 2x_2 = -3x_1$

or $4x_1 + 2x_2 = 0$

or $2x_1 + x_2 = 0$

$8x_1 + x_2 = -3x_2$

or $8x_1 + 4x_2 = 0$

or $2x_1 + x_2 = 0$

There is no unique soln, there should be infinite number of soln.

Let $x_1 = c$

$\therefore x_2 = -2c$

$$\psi = \begin{bmatrix} c \\ -2c \end{bmatrix}_{2\times 1}$$

using normalization condition (only for physics)

Rough:

$\bar{A}$ or A^* ⟶ Complex conjugate of the matrix

$$A = \begin{bmatrix} 2+i & -3i \\ -2 & 4i \end{bmatrix} \quad A^* \text{ or } \bar{A} = \begin{bmatrix} 2-i & 3i \\ -2 & -4i \end{bmatrix}$$

if ka sign sirf change hoga

dagger ←

$A^\dagger / A^H$ ⟶ complex conjugate transpose of the matrix

$$A^\dagger = (A^*)^T$$

$$A^\dagger = \begin{bmatrix} 2-i & -2 \\ 3i & -4i \end{bmatrix}$$

$\psi^\dagger\psi$ = total probability in Quantum mechanics

$\psi^\dagger\psi = 1$

or $\underset{1\times 2}{\begin{bmatrix} c & -2c \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} c \\ -2c \end{bmatrix}} = 1$

or $c^2 + 4c^2 = 1$

or $5c^2 = 1$

or $c = \pm\frac{1}{\sqrt{5}}$

normalised eigen vector,

$$\psi = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

For $\lambda_2 = 5$

$A\psi = \lambda I \psi$

or $\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

or $\begin{bmatrix} x_1 + 2x_2 \\ 8x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ 5x_2 \end{bmatrix}$

Now, $x_1 + 2x_2 = 5x_1$

or $2x_2 = 4x_1$

or $x_2 = 2x_1$

$8x_1 + x_2 = 5x_2$

or $8x_2 - 8x_1 = 4x_2$

or $x_2 = 2x_1$

Let, $x_1 = c$

$\therefore x_2 = 2c$

$\therefore \psi = \begin{bmatrix} c \\ 2c \end{bmatrix}$

Now, $\psi^+ \psi = 1$

$\Rightarrow \begin{bmatrix} c & 2c \end{bmatrix} \begin{bmatrix} c \\ 2c \end{bmatrix} = 1$

$\Rightarrow c^2 + 4c^2 = 1$

$\Rightarrow 5c^2 = 1$

$\Rightarrow c = \pm \frac{1}{\sqrt{5}}$

$\therefore$ normalised eigen vector

$$\psi = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(3) (i) Define Trace of a Matrix —

(ii) Write down two important properties of eigen value of a matrix.

— (i) Trace: The sum of elements or numbers along principal digonal of a square matrix is known as Trace.

Exm: $A = \begin{bmatrix} 1 & 4 & -2 \\ -6 & 4 & 2 \\ -5 & -2 & -2 \end{bmatrix}$ Trace of $A = 1 + 4 - 2 = 3$

(ii) Two important properties of eigen value of a matrix

(a) Sum of eigen value = Trace of the matrix

(b) Product of eigen v = Det of the matrix

Note: $A = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$ $\lambda_1 = -3$ $\lambda_2 = 5$

(i) $\lambda_1 + \lambda_2 = 2$ = Trace of the matrix

(ii) product of eigen value $= -15$ = det of the Matrix

$\det|A| = 1 - 16 = -15$

(8) Find the eigen value and eigen vector of the given matrix.

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$$

$$\therefore A - \lambda I = \begin{bmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{bmatrix}$$

Now,

$$\begin{vmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(1-\lambda)(-4-\lambda)-6\} + 3\{3(-4-\lambda)+15\} + 1[6+5(1-\lambda)] = 0$$

$$\Rightarrow -(2-\lambda)[(4+\lambda)(1-\lambda)+6] + 45 - 9(4+\lambda) + 1 + 5 - 5\lambda = 0$$

$$\Rightarrow (\lambda-2)[4-4\lambda+4-\lambda^2+1] + 45 - 31 - 9\lambda + 11 - 5\lambda = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 2\lambda = 0$$

$$\Rightarrow -\lambda(\lambda^2+\lambda-2) = 0$$

$$\lambda^2+\lambda-2 = 0$$

$$\Rightarrow \lambda^2 + 2\lambda - \lambda - 2 = 0$$

$$\Rightarrow \lambda(\lambda+2) - 1(\lambda+2) = 0$$

$$(\lambda-1)(\lambda+2) = 0$$

$\therefore \lambda_1 = 0$
$\lambda_2 = 1$
$\lambda_3 = -2$

2নং এ, $\lambda^3 - \lambda^2 - 2\lambda = 0$
তবে $\lambda - 1$ দিয়ে ভাগ করলে ভাগশেষ শূন্য হবে।
eigen value দুটি ভিন্ন same ও হতে পারে।

ATQ,

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 - 3x_2 + x_3 \\ 3x_1 + x_2 + 3x_3 \\ -5x_1 + 2x_2 - 4x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{bmatrix}$$

$2x_1 - 3x_2 + x_3 = -2x_1$

$\Rightarrow 4x_1 - 3x_2 + x_3 = 0$

$\Rightarrow 4x_1 - 3x_2 - x_1 - x_2 = 0$

$\Rightarrow 3x_1 - 4x_2 = 0$

$\Rightarrow 3x_1 = 4x_2$

$3x_1 + x_2 + 3x_3 = -2x_2$

$x_1 + x_2 + x_3 = 0$

$\Rightarrow x_3 = -(x_1 + x_2)$

$-5x_1 + 2x_2 - 4x_3 = -2x_3$

$-5x_1 + 2x_2 - 2x_3 = 0$

$-5x_1 + 2x_2 + 2x_1 + 2x_2 = 0$

$\Rightarrow -3x_1 + 4x_2 = 0$

$\Rightarrow 3x_1 = 4x_2$

Here (2) eqn and 3 variable.

Let $x_2 = c$ $\quad x_3 = -\left(c + \frac{4c}{3}\right) = -\frac{7c}{3}$

$x_1 = \frac{4}{3}c$

$$\psi = \begin{bmatrix} \frac{4}{3}c \\ c \\ -\frac{7c}{3} \end{bmatrix} \qquad \psi^\dagger = \begin{bmatrix} \frac{4}{3}c & c & -\frac{7c}{3} \end{bmatrix}$$

Using normalization condition, normalised eigenvector

$\psi^\dagger \psi = 1$

$$\begin{bmatrix} \frac{4}{3}c & c & -\frac{7c}{3} \end{bmatrix} \begin{bmatrix} \frac{4}{3}c \\ c \\ -\frac{7c}{3} \end{bmatrix} = 1$$

$\Rightarrow \frac{16}{9}c^2 + c^2 + \frac{49c^2}{9} = 1$

$\Rightarrow \frac{74c^2}{9} = 1$

$\Rightarrow c = \frac{3}{\sqrt{74}}$

$$\psi = \begin{bmatrix} \frac{4}{\sqrt{74}} \\ \frac{3}{\sqrt{74}} \\ -\frac{7}{\sqrt{74}} \end{bmatrix} = \frac{1}{\sqrt{74}} \begin{bmatrix} 4 \\ 3 \\ -7 \end{bmatrix}$$

$\rightarrow$ ATQ, **When $\lambda = 0$**

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 - 3x_2 + x_3 \\ 3x_1 + x_2 + 3x_3 \\ -5x_1 + 2x_2 - 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now,

$2x_1 - 3x_2 + x_3 = 0$

$\Rightarrow x_3 = 3x_2 - 2x_1$

$3x_1 + x_2 + 3x_3 = 0$

$\Rightarrow 3x_1 + x_2 + 9x_2 - 6x_1 = 0$

$\Rightarrow 10x_2 - 3x_1 = 0$

$-5x_1 + 2x_2 - 4x_3 = 0$

$\Rightarrow -5x_1 + 2x_2 - 12x_2 + 8x_1 = 0$

$\Rightarrow -10x_2 + 3x_1 = 0$

$\Rightarrow 10x_2 - 3x_1 = 0$

As here is two equation and 3 variable,

$x_1 = c_1$, $10x_2 = 3c_1$ $\quad x_3 = \frac{9c_1}{10} = 2c_1$

Now, $\quad x_2 = \frac{3c_1}{10} \quad = \frac{-11c_1}{10}$

$\psi^+ \psi = 1$

$$\begin{bmatrix} c_1 \\ \frac{3c_1}{10} \\ \frac{-11c_1}{10} \end{bmatrix} \begin{bmatrix} c_1 & \frac{3c_1}{10} & \frac{-11c_1}{10} \end{bmatrix} = \begin{bmatrix} c_1 \\ \frac{3c_1}{10} \\ \frac{-11c_1}{10} \end{bmatrix} = 1$$

$$c_1^2 + \frac{9c_1^2}{100} + \frac{121c_1^2}{100} = 1$$

$$\frac{131c_1^2}{100} = 1$$

$$c_1 = \sqrt{\frac{100}{131}} = \pm \frac{10}{\sqrt{131}}$$

$$\therefore \psi = \begin{bmatrix} \frac{10}{\sqrt{131}} \\ \frac{30}{10\sqrt{131}} \\ \frac{-110}{10\sqrt{131}} \end{bmatrix}$$

$$= \frac{10}{\sqrt{131}} \begin{bmatrix} 1 \\ \frac{3}{10} \\ \frac{-11}{10} \end{bmatrix}$$

H/W

(8) Find the eigen value and eigen vector of following matrix, $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

= Done in Rough.

→ State Cayley-Hamilton theorem with example.

= Every square matrix satisfy it's own characteristic equation.

exm: $A = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$

Det. $A - \lambda I = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 - 16 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 15 = 0 \quad \ldots (i)$

equ^n(i) is known as characteristic equ^n of A.

Now $A^2 - 2A - 15I = 0$

L.H.S –

$= \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} - 15\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+16 & 2+2 \\ 8+8 & 16+1 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 16 & 2 \end{bmatrix} - \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$

$= \begin{bmatrix} 17 & 4 \\ 16 & 17 \end{bmatrix} - \begin{bmatrix} 17 & 4 \\ 16 & 17 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore \boxed{A^2 - 2A - 15I = 0}$

→ Find the inverse of following matrix by using cayley-Hamilton theorem.

= (i) $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 & -4 \\ -2 & 6 \end{bmatrix}$

= (i) Det. $A - \lambda I = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & -3 \\ -4 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(1-\lambda) - 12 = 0$

$\Rightarrow 2 - 2\lambda - \lambda + \lambda^2 - 12 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 10 = 0 \quad \ldots (i)$

Using Cayley Hamilton theorem.

$A^2 - 3A - 10I = 0$

$\Rightarrow A^2A^{-1} - 3AA^{-1} - 10A^{-1} = 0$ (multiplying A^{-1})

$\Rightarrow A - 3I - 10A^{-1} = 0 \quad [AA^{-1} = I]$

$\Rightarrow \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - 10A^{-1} = 0$

$\Rightarrow \frac{1}{10}\begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix} = A^{-1}$

(ii) $A = \begin{bmatrix} 3 & -4 \\ -2 & 6 \end{bmatrix}$

Det $A - \lambda I = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & -4 \\ -2 & 6-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)(6-\lambda) - 8 = 0$

$\Rightarrow 18 - 3\lambda - 6\lambda + \lambda^2 - 8 = 0$

$\Rightarrow \lambda^2 - 9\lambda + 10 = 0$

Using Cayley Hamilton theorem —

$A^2 - 9A + 10I = 0$

$\Rightarrow A^2A^{-1} - 9AA^{-1} + 10A^{-1} = 0$

$\Rightarrow A - 9I + 10A^{-1} = 0$

$\Rightarrow 10A^{-1} = 9I - A$

$\Rightarrow 10A^{-1} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & -4 \\ -2 & 6 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 2 & 3 \end{bmatrix}$

(Q) Define hermitian matrix with example.

(i) Show that eigen value of Hermitian matrix is always real.

(ii) Show that every square matrix can be written as sum of hermitian matrix and skew-Hermitian matrix.

(i) Square matrix A is called Hermitian if

$$A^\dagger = A$$

Ex: $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$A^\dagger = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = A$$

$\therefore A^\dagger = A$, A is hermitian matrix.

(ii) Let, A is a hermitian matrix,

$$A^\dagger = A \quad \text{.....(i)}$$

From eigen value equation,

$$A\psi = \lambda I \psi \quad \text{---(ii)}$$

λ = eigen value
ψ = eigen vector

From equation (ii)

$$(A\psi)^\dagger = (\lambda I \psi)^\dagger$$

$$\Rightarrow \psi^\dagger A^\dagger = \lambda^* (I\psi)^\dagger$$

$$\Rightarrow \psi^\dagger A = \lambda^* \psi^\dagger I^\dagger \quad \text{[by using equation (i)]}$$

$$\Rightarrow \psi^\dagger A = \lambda^* \psi^\dagger I \quad \text{---(iii)}$$

$(AB)^\dagger = B^\dagger A^\dagger$

λ^*

here $*$ → complex conjugate

$I^\dagger = I$

multiplying eq (iii) by ψ (from right side)

$$\psi^\dagger A \psi = \lambda^* \psi^\dagger I \psi$$

$$\Rightarrow \psi^\dagger \lambda I \psi = \lambda^* \psi^\dagger I \psi \quad \text{[using eq(ii)]}$$

$$\Rightarrow \lambda \psi^\dagger I \psi = \lambda^* \psi^\dagger I \psi$$

$$\Rightarrow \boxed{\lambda^* = \lambda}$$

$A \cdot 2B = 2A \cdot B$

§ also verify it

$\therefore$ eigen value of hermitian matrix is always real.

(iii) Let, A is a square matrix

$$A = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger)$$

$$\Rightarrow A = B + C$$

$$B^\dagger = \frac{1}{2}(A + A^\dagger)^\dagger = \frac{1}{2}(A^\dagger + A)$$

$$\Rightarrow B^\dagger = B \quad [B \text{ is hermitian}]$$

$$C^\dagger = \frac{1}{2}(A - A^\dagger)^\dagger = \frac{1}{2}(A^\dagger - A) = -C$$

$\therefore$ [C is skew / anti hermitian]

let,

$B = \frac{1}{2}(A + A^\dagger)$

$C = \frac{1}{2}(A - A^\dagger)$

$(A+B)^T = A^T + B^T$

$(A^T)^T = A$

$\therefore (A+B)^\dagger = A^\dagger + B^\dagger$

$(A^\dagger)^\dagger = A$

∴ every square matrix can be written as sum of hermitian matrix and skew hermitian matrix

(5) Show that eigen vectors corresponding to diff eigen value of hermitian matrix is orthogonal or normal to each other.

= Let, A is a hermitian matrix with eigen value λ_1 & λ_2 $(\lambda_1 \neq \lambda_2)$. ψ_1, ψ_2 are eigen vectors of λ_1 & λ_2 respectively.

$$A^\dagger = A \quad \text{---(i)} \quad (\dagger \rightarrow \text{Dagger})$$

From eigen value equation -

$$A\psi_1 = \lambda_1 I \psi_1 \quad \text{---(ii)}$$

$$A\psi_2 = \lambda_2 I \psi_2 \quad \text{---(iii)}$$

From eqn (ii)

$$(A\psi_1)^\dagger = (\lambda_1 I \psi_1)^\dagger$$

$$\Rightarrow \psi_1^\dagger A^\dagger = \lambda_1^* \psi_1^\dagger I^\dagger$$

$$\Rightarrow \psi_1^\dagger A^* = \lambda_1 \psi_1^\dagger I \quad \text{---(iv)}$$

$\lambda_1 = \lambda_1^*$ as A is hermitian As, $A^\dagger = A$

multiplying eqn (iv) by ψ_2;

$$\psi_1^\dagger A \psi_2 = \lambda_1 \psi_1^\dagger I \psi_2$$

$$\Rightarrow \psi_1^\dagger \lambda_2 I \psi_2 = \lambda_1 \psi_1^\dagger I \psi_2 \quad [\text{using eqn (iii)}]$$

$$\Rightarrow \lambda_2 \psi_1^\dagger I \psi_2 - \lambda_1 \psi_1^\dagger I \psi_2 = 0$$

$$\Rightarrow (\lambda_2 - \lambda_1) \psi_1^\dagger I \psi_2 = 0 \quad \text{if } (\lambda_2 \neq \lambda_1 \because \lambda_2 - \lambda_1 \neq 0)$$

$$\therefore \psi_1^\dagger I \psi_2 = 0$$

$$\Rightarrow \boxed{\psi_1^\dagger \psi_2 = 0} \quad \text{or} \quad \boxed{\psi_2^\dagger \psi_1 = 0}$$

(row & column matrix multiply to angular)

So, eigen vectors corresponding to diff eigen value of hermitian matrix is orthogonal or normal to each other.

(Q) (i) ~~Sho~~ Define skew hermitian matrix with example.
(ii) Show that eigen value of skew hermitian is zero or purely imaginary.

= (i) Square matrix A is called skew-hermitian if

$$A^\dagger = -A$$

example, $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$$A^\dagger = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -A$$

$\therefore A^\dagger = -A$ [A is skew-Hermitian]

(ii) Let, A is a skew-hermitian matrix.

$$A^\dagger = -A \quad \text{---(i)}$$

From eigen value equation —

$$A\psi = \lambda I \psi \quad \text{---(ii)}$$

From equation (i) —

$$(A\psi)^\dagger = (\lambda I \psi)^\dagger$$

$$\Rightarrow \psi^\dagger A^\dagger = \lambda^* \psi^\dagger I^\dagger$$

$$\Rightarrow \psi^\dagger (-A) = \lambda^* \psi^\dagger I \quad \text{---(iii)}$$

multiplying (iii) by ψ

$$\Rightarrow -\psi^\dagger A\psi = \lambda^* \psi^\dagger I \psi$$

$$\Rightarrow -\psi^\dagger (\lambda I \psi) = \lambda^* \psi^\dagger I \psi \quad \text{using equation (ii)}$$

$$\Rightarrow \lambda (-\psi^\dagger I \psi) = \lambda^* \psi^\dagger I \psi$$

$$\Rightarrow \boxed{-\lambda = \lambda^*}$$

exam:
$(2i)^* \longrightarrow -2i$
$(+3i)^* \longrightarrow -3i$
$(0)^* \longrightarrow 0$

So, that eigen value of skew-hermitian is zero or purely imaginary.

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(8) (i) Define unitary matrix. unitary matrix $= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
(ii) Show that eigen value of unitary matrix have modulus value 1.

=(i) square matrix A is called unitary matrix if, $\boxed{A^\dagger A = I}$

$\Rightarrow A^\dagger A A^{-1} = A^{-1}$ $[A\ AA^{-1} = I]$

$\Rightarrow A^\dagger = A^{-1}$

(ii) Let A is an unitary matrix

$A^\dagger A = I$ ---(i)

From eigen value equation —

$A\psi = \lambda I \psi$ ——(ii)

From equation (ii)

$(A\psi)^\dagger = (\lambda I \psi)^\dagger$

$\Rightarrow \psi^\dagger A^\dagger = \lambda^* \psi^\dagger I^\dagger$

$\Rightarrow \psi^\dagger A^\dagger = \lambda^* \psi^\dagger I$ ----(iii)

multiplying (iii) by A

$\Rightarrow \psi^\dagger A^\dagger A = \lambda^* \psi^\dagger A$

$\Rightarrow \psi^\dagger . I = \lambda^* \psi^\dagger A$ [using (i) equ ii] ——(iv)

multiplying eqn (iv) by ψ

$\Rightarrow \psi^\dagger \psi = \lambda^* \psi^\dagger A \psi$

$\Rightarrow \psi^\dagger \psi = \lambda^* \psi^\dagger (\lambda I \psi)$ [equ (ii)]

$\Rightarrow \psi^\dagger \psi = \lambda^* \lambda (\psi^\dagger \psi)$

$\Rightarrow \boxed{\lambda^* \lambda = 1} \longrightarrow |\lambda^2| = 1 \quad \therefore |\lambda| = 1$

Rough, $z = x + iy \quad z^* = x - iy$

$\therefore zz^* = (x - iy)(x + iy) = x^2 + y^2$

$|z| = \sqrt{x^2 + y^2}$

$z^* z = |z|^2$

(8) Define Symmetric & skew Symmetric matrix.
(ii) Show that every square matrix can be written as sum of Symmetric & skew-symmetric matrix.

(i) Square matrix A is called symmetric if $A^T = A$
and if $A^T = -A$ then A is skew-symmetric or anti-symmetric.

(ii) Let, A is a square matrix.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$\boxed{B = \frac{1}{2}(A + A^T), \quad C = \frac{1}{2}(A - A^T)}$$

$$\therefore A = B + C \quad \text{---(i)}$$

$$B^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = B \quad [B \text{ is Symmetric}]$$

$$C^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -C$$

[C is skew symmetric]

From equation (i) every square matrix can be written as sum of symmetric & skew-symmetric matrix.

Note:

Let, A is a 3×3 matrix, with eigen value, $\lambda_1, \lambda_2, \lambda_3$

(i) Eigen value of A^2 will be $\lambda_1^2, \lambda_2^2, \lambda_3^2$

(ii) Eigen value of A^{-1} " " $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

(iii) " " " A^T " " $\lambda_1, \lambda_2, \lambda_3$

(iv) ---------- A^n ---------- $\lambda_1^n, \lambda_2^n, \lambda_3^n$ ---

($n = -1, 1, 2, 3 \ldots$)